STOCHASTIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATION IN FINANCIAL MODELING

Miloslav Torda
Supervisor: RNDr. Beáta Stehlíková, PhD.

FMFI UK

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Objectives of project

- Go through the basic theory behind Itô calculus
- To use this theory in the pricing models of financial derivatives - specifically future contracts.

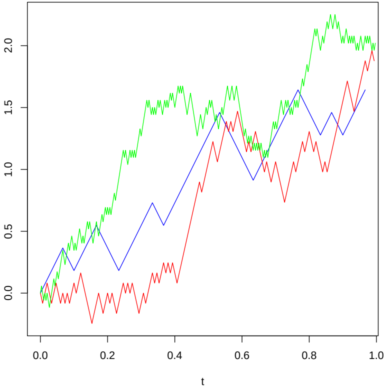
Brownian motion as a limit of a random walk

We define a stochastic process $X(t)$ in the following way

- $X(t) = \triangle x \left(X_1 + ... + X_{[\frac{t}{\triangle t}]} \right)$
- $X_i = \begin{cases} +1 & \text{if i-th step of length } \triangle x \text{ is to the right,} \\ -1 & \text{if it is to the left,} \end{cases}$
- $P[X_i = 1] = P[X_i = -1] = \frac{1}{2}$

i.e for every time unit $\triangle t$ we will make a step $\triangle x$ to the right or to the left with equal probability.

Example of a random walk with $\triangle x = \sigma\sqrt{\triangle t}$ and $\sigma = 1$



blue $\triangle t = \frac{1}{30}$, *red* $\triangle t = \frac{1}{150}$ *a* *green* $\triangle t = \frac{1}{300}$

Definition of brownian motion

We say that the stochastic process $\{X(t),\ t \geq 0\}$ is the process of Brownian motion

- $X(0) = 0$;
- $\{X(t), t \geq 0\}$ *has stationary and independent increments* i.e. the probability distribution $X(t+s) - X(t)$ does not depend on t and for every $t_1 < t_2 < ... < t_n$ are $X(t_1), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$ independent.
- for every $t > 0$ is $X(t)$ random variable from normal distribution with the expected value 0 and variance $\sigma^2 t$

If $\sigma = 1$, we say that the process is standardized Brownian motion.
Brownian motion with the coefficient of drift μ is when the expected value of $X(t)$ is μt

Existence and continuity of Brownian motion

- there exists a process with the properties from the previous slide
  - by defining a measure $\nu_{t_1,\dots,t_k}$ on $(\mathbb{R}^k, \mathcal{B})$ in the following way

$$\nu_{t_1,\dots,t_k}(F_1 \times \dots \times F_k) =$$

$$= \int\limits_{F_1 \times \dots \times F_n} f_{t_1}(x_1) f_{t_2-t_1}(x_2-x_1) \dots f_{t_k-t_{k-1}}(x_k-x_{k-1}) dx_1 \dots dx_k, \; F_i \in \mathbb{R}.$$

    where f_t is the density of normal distribution $\mathcal{N}(0,t)$
  - next, by applying one of Kolmogorov's theorems (*Kolmogorov extension theorem*)
- There exists a continuous version of Brownian motion *canonical Brownian motion (by using another Kolmogorov's theorem: Kolmogorov's continuity theorem)*

Definition of Itô's integral

Let $f \in \nu(S, T)$. Then Itô's integral f (from S to T) is defined in the following way

$$\int_S^T f(t,\omega)dB_t(\omega) := lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega)$$

(the limit in $L^2(P)$), where $\{\phi_n\}$ is the sequence of simple functions with the following property

$$E\left[\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 dt\right] \to 0$$

as $n \to \infty$.

Properties of Itô calculus

Let $f, g \in \nu(0, T)$ and $0 \leq S < U < T$. Then

- $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$ for almost every ω
- $\int_S^T (cf + g) dB_t = c \int_S^T f dB_t + \int_S^T g dB_t$ for almost every ω
- $E\left[\int_S^T f dB_t\right] = 0$
- $\int_S^T f dB_t$ je $\mathcal{F}_t$-measurable
- (Itô isometry) $E\left[\left(\int_S^T f(t,\omega) dB_t\right)^2\right] = E\left[\int_S^T f(t,\omega)^2 dt\right]$

Itô lemma

Let X_t be Itô process given by

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$ (i.e. g is twice differenentiable function on $[0,\infty) \times \mathbb{R}$). Then

$$Y_t = g(t, X_t)$$

is again Itô process and

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{\partial^2 g}{\partial x^2}(t, X_t) \cdot (dX_t)^2,$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is derived according to the following rules

$$dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0, \; dB_t \cdot dB_t = dt$$

Stochastic differential equation

A stochastic differential equation is an equation of the form:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t.$$

with the initial condition $X_0 = x$ and $x \in \mathbb{R}$.

We call functions $b(t, x)$ and $\sigma(t, x)$ *drift* a *diffusion*.

Geometric Brownian motion

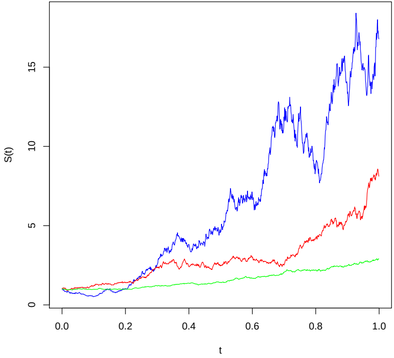
- Is the following stochastic differential equation:

$$dS(t) = rS(t)dt + \alpha S(t)dB(t),$$

  r and α are constants
- Solution

$$S(t) = S(0) \exp\left[(r - \frac{\alpha^2}{2})t + \alpha B(t)\right]$$

Examples of geometric Brownian motion with the initial condition $S(0) = 1$



blue $r = 2.1$, $\alpha = 0.9$, *red* $r = 1.7$, $\alpha = 0.6$ *a* *green* $r = 1.1$, $\alpha = 0.3$.

Properties of geometric brownian motion

- $S(t)$ is from log-normal distribution
- $E[S(t)] = S(0)\exp(rt)$ - from the properties of log-normal distribution
- or more generally from Itô lemma

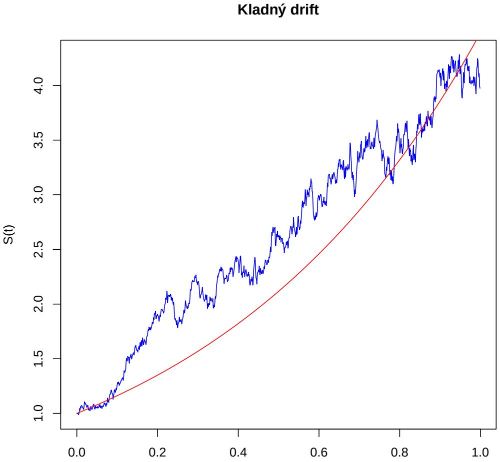
$$E[S(t)^{\beta}] = S(0)^{\beta}\exp\left[\beta\left(r - \frac{\alpha^2}{2}\right)t + \frac{\beta^2\alpha^2}{2}t\right]$$

for $\beta \in \mathbb{R}$

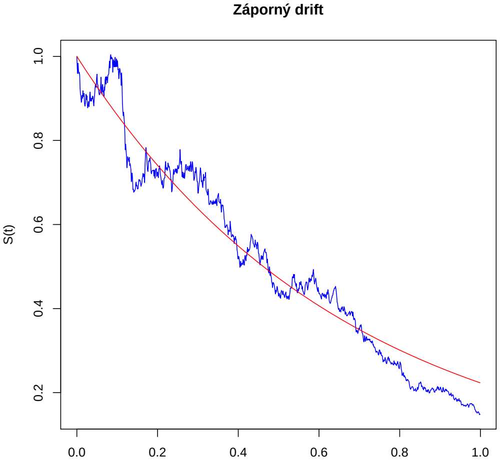
- $100(1-\beta)\%$-th two-sided confidence interval

$$P\left(S(0)e^{\left(r-\frac{\alpha^2}{2}\right)t-u\left(\frac{\beta}{2}\right)\sqrt{\alpha^2 t}} \leq S(t) \leq S(0)e^{\left(r-\frac{\alpha^2}{2}\right)t+u\left(\frac{\beta}{2}\right)\sqrt{\alpha^2 t}}\right) = 1-\beta.$$

Example of expected value of geometric Brownian motion with positive drift



Example of expected value of geometric Brownian motion with negative drift

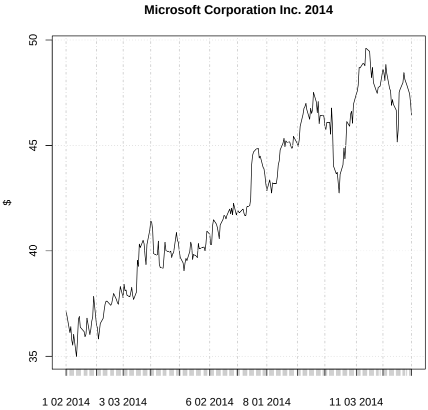


Geometric Brownian motion and stock price modeling

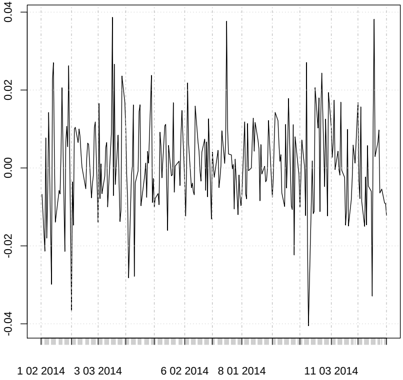
Assumptions:

- Rates of price changes $\ln(\frac{S_t}{S_{t-1}})$ are independent
  - we can check this by autocorrelation and Ljung-Box test.
- $\ln\left(\frac{S_t}{S_{t-1}}\right)$ are from normal distribution
  - Kolmogorov - Smirnov: p-value = 0.535

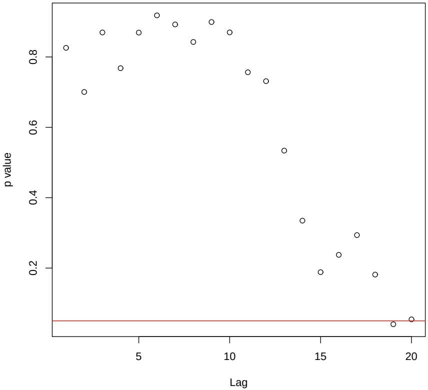
Daily close prices of Microsoft Corporation Inc. in year 2014



$Ln\left(\frac{S_t}{S_{t-1}}\right)$, where S_t is the close price of a stock at time t

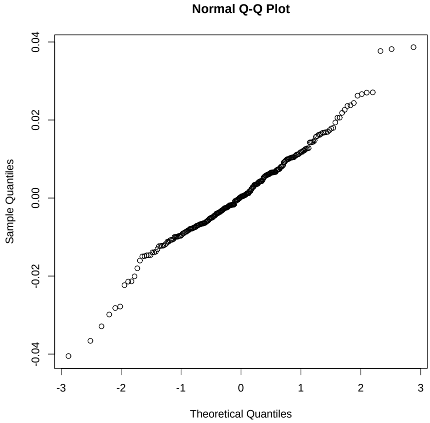


p-values of Ljung-Box test for different values of lag

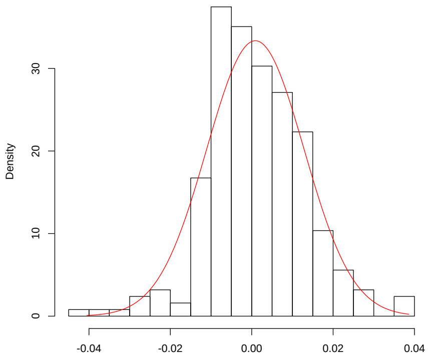


— *denotes p-value* $= 0.05$.

Q-Q plot of $\ln\left(\frac{S_t}{S_{t-1}}\right)$



Comparison of histogram of $\ln\left(\frac{S_t}{S_{t-1}}\right)$ and normal distribution density



Estimating parameters of geometric Brownian motion

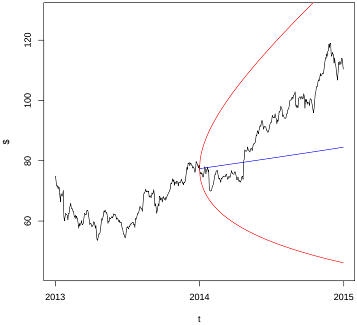
Discretization of $S(t)$

$$\ln\left(\frac{S(t)}{S(t-\triangle t)}\right) = \left(r - \frac{\alpha^2}{2}\right)\triangle t + \alpha(B(t) - B(t-\triangle t))$$

and from the properties of normal distribution we get the estimators

$$\hat{r} = \frac{m}{\triangle t} + \frac{\hat{\alpha}^2}{2}, \ \hat{\alpha} = \sqrt{\frac{s^2}{\triangle t}}$$

Example of using estimation of parameters of geometric Brownian motion on the prices of Apple Inc. in 2014



— expected value of GBP from the estimated parameters and — is 95%-th pointwise confidence band.

Ornstein - Uhlenbeck proces

- Stochastic differential equation in the form

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t$$

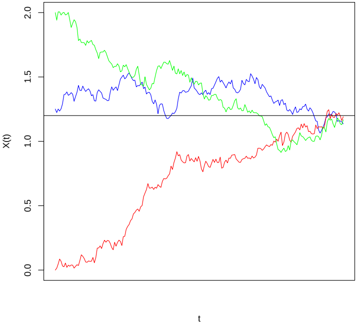
with the initial condition X_0, where $\theta > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants and B_t is Brownian motion.

- Solution

$$X_t = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) + \int_0^t \sigma e^{-\theta(t-s)} dB_s, \; 0 \leq t < \infty$$

- The process $\{X(t), \; t \geq 0\}$ is know as Ornstein - Uhlenbeck process.

Three examples of Ornstein-Uhlenbeck process with parameter values $\theta = 1$, $\mu = 1.2$, $\sigma = 0.3$ and various initial conditions



blue $X(0) \sim N(\mu, \frac{\sigma^2}{2\theta})$, *red* $X(0) = 0$ *a green* $X(0) = 2$. — *denotes* μ.

Properties of Ornstein - Uhlenbeck process

- $X(t)$ is from normal distribution
- From the properties of Itô integral we get

$$E[X_t|X_0] = X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}).$$

$$Var[X_t|X_0] = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})$$

- mean-reverting property i.e. pre $t \to \infty$

$$X_\infty \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{2\theta}\right).$$

- $100(1-\alpha)\%$-th confidence interval

$$P[-u\left(\frac{\alpha}{2}\right)\sqrt{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})} + X_0 e^{-\theta t} + \mu(1 - e^{-\theta t}) \leq X_t \leq$$

$$\leq u\left(\frac{\alpha}{2}\right)\sqrt{\frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})} + X_0 e^{-\theta t} + \mu(1 - e^{-\theta t})] = 1 - \alpha$$

Parameter estimation of Ornstein-Uhlenbeck process

From the conditional density of $X_{t_i}|X_{t_{i-1}}$

$$f(x_{t_i}|x_{t_{i-1}}) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\left[-\frac{x_{t_i} - x_{t_{i-1}}e^{-\theta(t_i - t_{i-1})} - \mu(1 - e^{-\theta(t_i - t_{i-1})})}{2\tilde{\sigma}^2}\right],$$

$$\tilde{\sigma}^2 = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta(t_i - t_{i-1})})$$

and from maximum likelihood estimation method we get the estimators

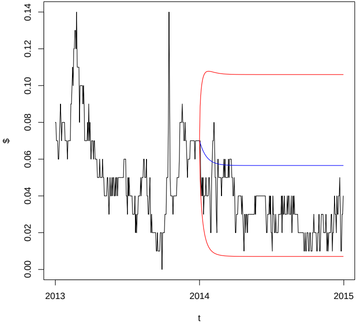
$$\widehat{\mu} = \frac{\sum_{i=1}^{n} x_{t_i} \sum_{i=1}^{n} x_{t_{i-1}}^2 - \sum_{i=1}^{n} x_{t_{i-1}} \sum_{i=1}^{n} x_{t_i} x_{t_{i-1}}.}{n(\sum_{i=1}^{n} x_{t_{i-1}}^2 - \sum_{i=1}^{n} x_{t_i} x_{t_{i-1}}) - [(\sum_{i=1}^{n} x_{t_{i-1}})^2 - \sum_{i=1}^{n} x_{t_{i-1}} \sum_{i=1}^{n} x_{t_i}]}$$

$$\widehat{\theta} = -\frac{1}{\triangle t} \ln\left[\frac{\sum_{i=1}^{n}(x_{t_i} - \widehat{\mu})(x_{t_{i-1}} - \widehat{\mu})}{\sum_{i=1}^{n}(x_{t_{i-1}} - \widehat{\mu})^2}\right]$$

$$\widehat{\tilde{\sigma}^2} = \frac{1}{n}\sum_{i=1}^{n}[x_{t_i} - x_{t_{i-1}}e^{-\widehat{\theta}\triangle t} - \widehat{\mu}(1 - e^{-\widehat{\theta}\triangle t})]^2$$

$$\widehat{\sigma^2} = \widehat{\tilde{\sigma}^2}\frac{2\widehat{\theta}}{1 - e^{-2\widehat{\theta}\triangle t}}.$$

Example of using parameter estimation for modeling daily prices of 3-month T-Bills in the year 2013



— is expected value of Orstein-Uhlenbeck process with the estimated parameters and — is 95%-th pointwise confidence interval.

Risk neutral pricing formula na future contracts

- Risk neutral pricing formula:

$$V(t) = \widetilde{E}[e^{-\int_t^T R(u)\mathrm{d}u} V(T)|\mathcal{F}(t)],\ 0 \leq t \leq T.$$

  $V(t)$ denotes the price of financial derivative at time t.
- Future price of an asset, the value of which at time T is $S(T)$, is given by the formula

$$F(t,T) = \widetilde{E}[S(T)|\mathcal{F}(t)],\ 0 \leq t \leq T.$$

Model 1

- Let the stochastic process of the spot price of a commodity be given by

$$\mathrm{d}S = \kappa(\mu - \ln S)S\mathrm{d}t + \sigma S\mathrm{d}B.$$

- Substitution $X = \ln S$ and transforming the equation in relation to risk neutral measure we get:

$$\mathrm{d}X = \kappa(\alpha^* - X)\mathrm{d}t + \sigma \mathrm{d}B^*$$

- From the properties of Ornstein-Uhlenbeck process and from deffinition of future contracts we get the price of the future contract at time t:

$$F(S,t) = \widetilde{E}\left[S(t)\right] =$$

$$= \exp\left(\ln S(t) e^{-\kappa(T-t)} + \alpha^*(1 - e^{-\kappa(T-t)}) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(T-t)})\right)$$

Model 2

- Suppose that the spot price of the commodity follows the process
$$dS_t = \kappa(\alpha - S_t)\,dt + \sigma(S_t)^\gamma\,dB_t$$
for some risk neutral measure
- By solving this stochastic differential equation and from the properties of Itô integral we get the price of the future contract:

$$F(S,t) = \widetilde{E}\,[S_t] = e^{-\kappa(T-t)}[S_t - \alpha(1 - e^{\kappa(T-t)})]$$

Model 3

- Two factor model

$$dS_t = \mu S_t dt + \sigma_1 S_t dW_t,$$

$$d\delta_t = \kappa(\alpha - \delta_t)dt + \sigma_2 dZ_t,$$

  - S_t is the spot price of the commodity
  - δ_t is *convenience yield*
  - dW_t and dZ_t are the increments of the standardized Brownian motion with correlation

$$dW_t dZ_t = \rho dt.$$

- From the theory of risk neutral pricing we have derived the price of the future contract.

Price of the future contract for model 3

$$
\begin{aligned}
F(T-t,S,\delta) = S(t)\exp\{[(-\alpha + \frac{1}{\kappa}(\sigma_2\lambda - \sigma_1\sigma_2\rho + r) + \frac{\sigma_2^2}{2\kappa^2})](T-t) \\
- \frac{1}{\kappa}[\delta_t - \alpha + \frac{1}{\kappa}(\sigma_2\lambda - \sigma_1\sigma_2\rho) + \frac{\sigma_2^2}{\kappa^2}](1-e^{-\kappa(T-t)}) \\
+ \left(\frac{\sigma_2}{\kappa}\right)^2 \frac{1}{4\kappa}(1-e^{-2\kappa(T-t)})\}.
\end{aligned}
$$

Parameter estimation for model 1

- estimation of parameters κ and σ using estimators derived in the Ornstein - Uhlenbeck process section
- As for estimation of parameter α^* we minimized the function

$$f(\hat{\alpha^*}) = \sum_{i=1}^{N} \left[\ln F_i - \ln S_i e^{-\kappa(T_i - t_i)} - \alpha^*(1 - e^{-\kappa(T_i - t_i)}) - \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(T_i - t_i)}) \right]^2$$

- Estimator is then α^*:

$$\hat{\alpha^*} = \frac{\sum_{i=1}^{N} \left[\ln F_i - \ln S_i e^{-\kappa(T_i - t_i)} - \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa(T_i - t_i)}) \right] (1 - e^{-\kappa(T_i - t_i)})}{\sum_{i=1}^{N} (1 - e^{-\kappa(T_i - t_i)})^2}.$$

Parameter estimation for model 2

- Linear regression implemented in R environment applied on discretization of model 2

$$S_{t_{i+1}} - S_{t_i} = \kappa \left(\alpha - S_{t_i}\right) \triangle t + \sigma \left(S_{t_i}\right)^{\gamma} \epsilon_{t_i}$$

for $\gamma = \frac{1}{2}$.

Parameter estimation for model 3

- Estimators of parameters κ, α, σ_1, σ_2 and ρ were found using SUR model (seemingly unrelated regression model) applied on discretized approximation of model 3

$$\delta_t - \delta_{t-1} = \alpha k + k\delta_{t-1} + \varepsilon_t$$

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = a + \varepsilon_t.$$

- To estimate λ we applied least squares on function

$$f(\lambda) = \sum_{i=1}^{n}\{\ln F_i - \ln S_i - [(-\alpha + \frac{1}{\kappa}(\sigma_2\lambda - \sigma_1\sigma_2\rho + r) + \frac{\sigma_2^2}{2\kappa^2})](T_i - t_i)$$
$$+ \frac{1}{\kappa}[\delta_t - \alpha + \frac{1}{\kappa}(\sigma_2\lambda - \sigma_1\sigma_2\rho) + \frac{\sigma_2^2}{\kappa^2}](1 - e^{-\kappa(T_i - t_i)})$$
$$- \left(\frac{\sigma_2}{\kappa}\right)^2 \frac{1}{4\kappa}(1 - e^{-2\kappa(T_i - t_i)})\}^2$$

Parameter estimation for model 3

$$\hat{\lambda} = \frac{\sum_{i=1}^{n}\{lnF_i - lnS_i - [(-\alpha + \frac{1}{\kappa}(-\sigma_1\sigma_2\rho + r) + \frac{\sigma_2^2}{2\kappa^2})](T_i - t_i)\}(T_i - t_i - 1 + e^{\kappa(T_i - t_i)})}{\sum_{i=1}^{n}\frac{\sigma_2}{\kappa}(T_i - t_i - 1 + e^{\kappa(T_i - t_i)})^2}$$
$$+ \frac{\sum_{i=1}^{n}\{\frac{1}{\kappa}[\delta_t - \alpha + \frac{1}{\kappa}(\sigma_2\lambda - \sigma_1\sigma_2\rho) + \frac{\sigma_2^2}{\kappa^2}](1 - e^{-\kappa(T_i - t_i)})\}(T_i - t_i - 1 + e^{\kappa(T_i - t_i)})}{\sum_{i=1}^{n}\frac{\sigma_2}{\kappa}(T_i - t_i - 1 + e^{\kappa(T_i - t_i)})^2}$$
$$- \frac{\sum_{i=1}^{n}\{(\frac{\sigma_2}{\kappa})^2\frac{1}{4\kappa}(1 - e^{-2\kappa(T_i - t_i)})\}(T_i - t_i - 1 + e^{\kappa(T_i - t_i)})}{\sum_{i=1}^{n}\frac{\sigma_2}{\kappa}(T_i - t_i - 1 + e^{\kappa(T_i - t_i)})^2}.$$

Applying models to real world data

- Data: daily observations of prices of futures of corn traded on Chicago Mercantile Exchange and Chicago Board of Trade.
- Estimation of parameters of models: daily data of 6 contracts during the year 2013 (march 2013, may 2013, july 2013, september 2013, december 2013, march 2014)
- Prediction comparison: 6 future contracts during the year 2014 (march 2014, may 2014, july 2014, september 2014, december 2014, march 2015)
- Computation of *convenience yield*

$$\delta = r_{T-3,T} - \frac{1}{\frac{3}{12}} \times ln\left[\frac{F(S,T)}{F(S,T-3)}\right]$$

- Computation of $r_{T-3,T}$

$$r_{i,j} = \left[\frac{(1+r_j)^j}{(1+r_i)^i}\right]^{\frac{1}{j-i}}$$

where r_j is daily rate of a Treasury bill with expiry time j

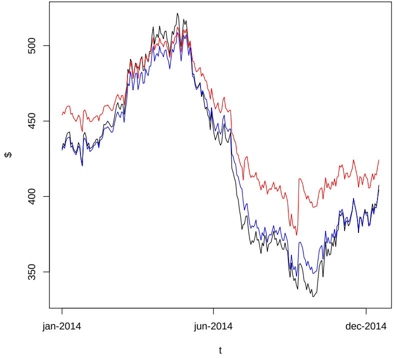
Results for model 1

	Our estimation	optim()
α^*	6.1504853	6.127819e+00
θ	0.9516584	4.143069e-01
σ^2	0.1477166	2.754954e-09
RMSE_1	28.70621	8.251041
RMSE_2	0.05789166	0.01991426

$$\text{RMSE}_1 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{F}_i - F_i)^2}$$

$$\text{RMSE}_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[\ln(\hat{F}_i) - \ln(F_i)]^2}$$

Model 1



black real data of corn price, red predictions based on our estimators, blue predictions based on parameter estimation using optim() function in R enviroment

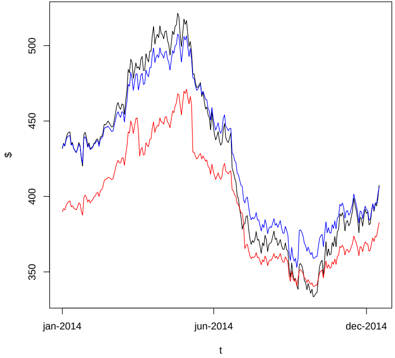
Result for model 2

	Linear regression	optim()
α	357.2541	458.2553718
κ	1.188226	0.5010952
RMSE_1	33.359	11.25532
RMSE_2	0.07366489	0.02836555

$$\text{RMSE}_1 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{F}_i - F_i)^2}$$

$$\text{RMSE}_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[\ln(\hat{F}_i) - \ln(F_i)]^2}$$

Model 2



black real corn price data, red predictions based on parameter estimation using linear regression, blue predictions based on parameter estimation using optim() function in R enviroment

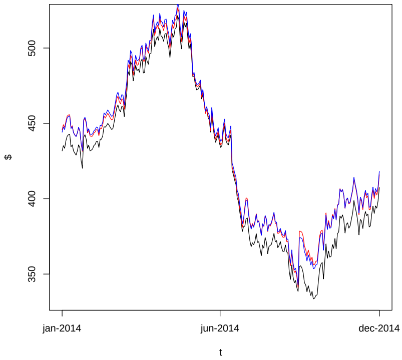
Results for model 3

	SUR and LS	optim()
α	0.13051337	0.131009916
κ	0.01066044	0.005171215
ρ	0.59788300	0.597961434
λ	-1.90872920	-1.911964878
σ_1	0.02422901	0.026163569
σ_2	0.06216214	0.158931695
RMSE_1	11.55318	11.80623
RMSE_2	0.03005018	0.03005476

$$\text{RMSE}_1 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{F}_i - F_i)^2}$$

$$\text{RMSE}_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[\ln(\hat{F}_i) - \ln(F_i)]^2}$$

Model 3



black real corn price data, red predictions based on SUR and LS paramater estimation, blue predictions based on parameter estimation using optim() function in R enviroment

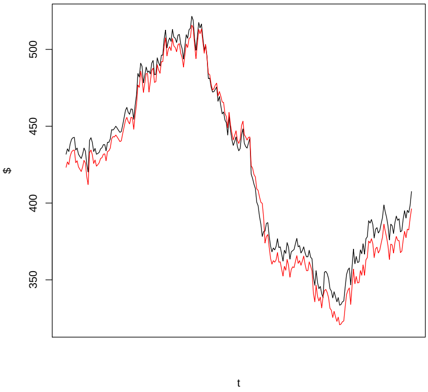
Trivial model

$RMSE_1$	8.962294
$RMSE_2$	0.02353489

$$\mathrm{RMSE}_1 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(F_i - S_i)^2}$$

$$\mathrm{RMSE}_2 = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[\ln(F_i) - \ln(S_i)]^2}$$

Trivial model



black real corn price data, red predictions based on trivial model

Feynman-Kac theorem

Consider a stochastic differential equation

$$\mathrm{d}X(u) = \beta(u, X(u))\mathrm{d}u + \gamma(u, X(u))\mathrm{d}W(u).$$

Let $h(y)$ be Borel-measurable function. Fix $T > 0$, and let $t \in [0, T]$ be given. We define a function

$$g(t,x) = E^{t,x}h(X(T)).$$

(We assume, that $E^{t,x}|h(X(T))| < \infty$ for every t and x.) Then $g(t,x)$ satisfies the parcial differential equation

$$g_t(t,x) + \beta(t,x)g_x(t,x) + \frac{1}{2}\gamma^2(t,x)g_{xx}(t,x) = 0$$

with the condition

$$g(T,x) = h(x) \text{ for every x.}$$

An alternative method of solving the future contract price

- Expression of model 1 relative to risk neutral measure $\widetilde{P}$

$$dX = \kappa(\alpha^* - X)dt + \sigma dB^*$$

  where $X = \ln S$
- The price of the future contract at time t:

$$F(t, S) = \widetilde{E}[S(t)] = \widetilde{E}[e^X]$$

- Directly from Feynman-Kac theorem we get the parcial differential equation

$$F_t + \kappa(\alpha^* - x)F_x + \frac{1}{2}\sigma^2 F_{xx} = 0$$

  with the condition

$$F(T, x) = e^x$$

- solving this quation we get the solution of model 1